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Homotopic embeddings and stably isotopic embeddings in a 1-connected smooth 4-manifold

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§ Introduction

We shall give some sufficient conditions that two embeddings of S^2 in a 1-connected 4-manifold M are isotopic in the stabilized manifold $M\#(\#S^2\times S^2)$.

THEOREM 1. Let M be a smooth 1-connected closed 4-manifold. Let S_0^2 and S_1^2 be 2-spheres smoothly embedded in M. Suppose that $S_0^2 \simeq S_1^2$, $\pi_1(M-S_0^2) = \pi_1(M-S_1^2) = 0$ and $[S_0^2]^2 = 0$. Then, S_0^2 and S_1^2 are isotopic in $M\#(\underset{n}{\#}S^2\times S^2)$ for some $n \geq 0$.

THEOREM 2. Let M be a smooth 1-connected closed 4-manifold and S^2 a smoothly embedded 2-sphere in M. Suppose that $\pi_1(M-S^2) \cong Z$ and $S^2 \simeq 0$ in M. Then, S^2 is unknotted in $M\#(\underset{n}{\#}S^2\times S^2)$ for some $n \geq 0$.

Theorem 1 is easily generalized to the case of the finite collection of pairs of embedded 2-spheres.

If $M= S^4$, the result of Theorem 2 implies that $S^4 - S^2 \simeq S^1$ be the argument at the end of §3 in [ ]. Hence, we get a geometric proof of the result of Kawauchi that $\pi_1(S^4-S^2) \cong Z$ implies $S^4 - S^2 \simeq S^1$.

§ 1 Proof of Theorem 1 (Outline)

We know that S_0^2 and S_1^2 are regular homotopic in M. But it seems not to help us much. The point of the proof is to find a reconstruction of S_1^2 in $M\#(\#S^2 \times S^2)$ as a ribbon sum of the copies of S_0^2, the first component S^2 of each $S^2 \times S^2$ and the trivial knots $S^2 \subset D^4$. We use the 5-dimensional surgery theory and the 5-dimensional Cerf theory for this purpose. If the ribbons could be deformed into the standard places, it is easy to understand that S_1^2 and S_0^2 are isotopic. In order to do this we use the Casson's trick which makes $\pi_1(M-S_1^2 \cup S_0^2) \cong Z$ by an isotopic deformation of S_0^2. Then, the difference between the actual ribbon and the virtual standard ribbon is S^1 embedded in $M\#(\#S^2 \times S^2) - S_1^2 \cup S_0^2$. It is easy to choose S^1 so that $S^1 \simeq 0$ outside $S_1^2 \cup S_0^2$. Hence, the surgery along these S^1 changes $M\#(\#S^2 \times S^2) - S_0^2 \cup S_1^2$ into $(M\#(\#S^2 \times S^2) - S_0^2 \cup S_1^2)\ \#$ $(S^2 \times S^2)$ and makes the actual ribbons isotopic to the virtual standard ribbons in the ambient manifold. (To be completed.)

§ 2 Proof of Theorem 2

The proof is easy and standard. Since $S^2 \simeq 0$ in M, we have $S^2 \times D^2 \subset M$. And $* \times \partial D^2 \subset M - S^2$ gives a generator of $\pi_1(M - S^2) \cong Z$. This implies that there exists a map $f: M - S^2 \times \mathring{D}^2 \to S^1$ which is an extension of the projection : $S^2 \times \partial D^2 \to S^1$. We make f transversely regular at a point of S^1 and get a connected smooth 3-manifold $N \subset M$ such that $\partial N = S^2$ in M.

In case M has a spin structure, we can restrict the spin structure of M on N and extend it over $N \cup D^3$, because the spin structure is determined by a framing of the stable tangent bundle over the 2-skelton (cf. Milnor [ ]). Since the 3-dimensional spin cobordism group vanishes [ibid], we have a smooth spin cobordism $(W^4; N^3, D^3)$ relative boundary. We may assume that W^4 is the union of the elementary cobordisms consisting of one of 1-handles, 2-handles and 3-handles in this order. The elementary cobordism $N \times I \cup$ (1-handle) is easily embedded in M and the spin structure on the other boundary is compatible with that of M. By an inductive argument on the number of 1-handles, the level manifold N_1 just above all the 1-handles is embedded in M and $\partial N_1 = S^2$. Remark that the spin structure of $N_1 \subset W$ is compatible with that of $N_1 \subset M$. The elementary cobordism $N_1 \times I \cup$ (2-handle) cannot be embedded in M but can be embedded in $M \# S^2 \times S^2$. In fact, we take $S^1 \subset N_1$ which is the boundary of the axis of the 2-handle. Then, $S^1 \simeq 0$ in $M - S^2$, because S^1 does not link with S^2 and $\pi_1(M - S^2) \cong$

The framing of $S^1 \times D^3$ is uniquely determined by the spin structure of W and the surgyry along this $S^1 \times D^3$ changes $M - S^2$ into $(M - S^2)\# S^2 \times S^2$. Note that we do not get $(M - S^2)\# S^2 \tilde{\times} S^2$ because of the choice of the spin structure. Of course, the spin structure on the other boundary is compatible with that of $M\# S^2 \times S^2$, because $M\# S^2 \times S^2$ has a unique spin structure from the fact that $H^1(M\# S^2 \times S^2 : Z_2) = 0$. The level manifold N_2 just above all the 2-handles is embedded in $M\#(\#_k S^2 \times S^2)$ and $\partial N_2 = S^2$, where k is equal to the number of the 2-handles of (W, N). We note that there is a diffeomorphism $h : (\#_\ell S^1 \times S^2 - \mathring{D}^3, \partial) \to (N_2, \partial)$, where ℓ is the number of 3-handles of (W, N) i.e., 1-handles of (W, D^3). Take the component S^1 of $S^1 \times S^2$ and consider $h(S^1) \subset N_2 \subset M\#(\#_k S^2 \times S^2)$. As before, $h(S^1) \simeq 0$ in $M\#(\#_k S^2 \times S^2) - S^2$. We can choose a framing of the tubular neighborhood of $h(S^1)$ so that the surgery along $h(S^1)$ changes $M\#(\#_k S^2 \times S^2) - S^2$ into $(M\#(\#_k S^2 \times S^2) - S^2)\# S^2 \times S^2$. Then $N_2' \cong (\#_{\ell-1} S^1 \times S^2 - \mathring{D}^3)$ is easily embedded in $M\#(\#_{k+1} S^2 \times S^2)$ such that $\partial N_2' = S^2$. By the induction we get a smooth submanifold N_3 of $M\#(\#_{k+\ell} S^2 \times S^2)$ such that $\partial N_3 = S^2$ and N_3 is diffeomorphic to D^3. This means that S^2 is unknotted in $M\#(\# S^2 \times S^2)$.

In the other case that $w_2(M) \neq 0$, we have only to remark that the surgery along the trivial circle with any framing gives us $M \# S^2 \times S^2$. Since the closed 3-manifold $N \cup D^3$ is orientabe and the tangent bundle is trivial, there is a spin structure on $N \cup D^3$ and any choice of the spin structure on $N \cup D^3$ leads to the same proof as above. q. e. d.

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